General charge conjugation operators in simple Lie groups

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A description of particular elements ("charge conjugation operators") found in any compact simple Lie group K is presented. Such elements R_i transform a physical state (weight vector of a basis of a representation space) into others with opposite "charge" (ith component of the weight), sometime changing also the sign of the state. It is demonstrated that exploitation of these elements and the finite subgroup N of K generated by them offer new powerful methods for computing with representations of the Lie group. Their application to construction of bases in representation spaces is considered in detail. It represents a completely new direction to the problem.

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I. INTRODUCTION

In this article we study certain elements R of order four, i.e., $R^4 = 1$, in connected compact simple Lie groups in order to demonstrate that they provide a new and powerful tool for applications. Although their importance has long been understood in the theory of Lie groups,[1] these elements have so far not been used in physics literature except for Refs. 2 and 3 (which are based entirely on this work) and they appear here for the first time in what might be called the theory of computation with Lie groups.

Intuitively these elements can be viewed in the following way: Given a simple Lie group K of rank l, then in a description of relevant physical states $|\lambda_1\lambda_2\cdots\lambda_l\rangle$, which are weight vectors in a representation space V of K, an important role is played by "quantum numbers" or "charges" λ_i, $i = 1,2...,l$, which are defined as eigenvalues of l suitably chosen linearly independent "diagonal" elements of the Lie algebra of K. The subject of our article is the elements R_i, $i = 1,2,...,l$, of K which permute the weight vectors of the same K-multiplet in such a way that $\lambda_i \rightarrow -\lambda_i$ if $\lambda_i \neq 0$. For lack of any better name we call R_i the charge conjugation operators (CCO) although it is only in special situations that one of them may coincide with the usual operator reversing electric charge.[4] It turns out that the action of R_i on $|\lambda_1\lambda_2\cdots\lambda_l\rangle$ is quite nontrivial. Besides reversing the charges (components of weights) they sometimes reverse the sign of the state or permute several states with the same "quantum numbers" (weights) when $\lambda_i = 0$. Let us underline the fact that there are no charge conjugating elements in K which would be of order 2 in all finite-dimensional representations of K.

The role which R_i may play in applications far exceeds the charge conjugation. In that respect Refs. 2 and 3, where they provide the main tool of the approach, are only modest illustrations of the possibilities. There all nonzero Clebsch–Gordon coefficients arising in a tensor product of two irreducible representation spaces of K are given by a small representative subset of them and any other coefficient is identified with one of the subset using CCO. Fortunately, the economy made this way rapidly increases with the rank l of K roughly being proportional to the order $|W|$ of the Weyl group W of K. Thus for instance, in the case of rank one group SU(2), the saving made by using CCO is the smallest because $|W| = 2$. It is equivalent to the well-known fact that from each pair of SU(2) Clebsch–Gordon coefficients $C(l_1,l_2l;\ m_2m_2m)$ and $C(l_1l_2l;\ -m_1\ -m_2\ -m)$ it suffices to calculate only one of them. However, for SU(n) the economy provided by CCO increases as $n!$.

In particle physics it is conceivable that the usual requirements of invariance of a (grand unified) model under the action of a reductive Lie subgroup K' of a semisimple group K is too strong and that all the conclusions drawn from the model would follow requiring only the invariance under the action of a finite subgroup F of K generated by R_i and possibly some other elements of finite order in K. In general, the finite subgroup N of K generated by R_i is of importance whenever K appears, even if its role so far has not been fully appreciated.

So far the possibilities of building N-invariant models which are not K'-invariant remain completely unexplored. They would closely resemble the K'-invariant ones in that the K-multiplets would be formed as direct sums of N-multiplets, but they would be simpler because N as a finite group has only finitely many irreducible representations.

Questions of this type motivate our undertaking although we do not address them directly in the article.

The first objective of the paper is to bring together what is known about CCO in a coherent way.

The second objective is related to the problem of construction of bases in representation spaces. Until now, in spite of the obvious importance of the problem, there is no satisfactory general method of construction. [*Note added in proof*: Daya-Nand Verma has produced an as yet unproved algorithm for constructing bases by the first of the methods below. It appears very promising.] Indeed, there are three well-known ways how to construct a basis. The first is a multiple application of generators to one basis vector. Although this is a general method in principle, practically it is so unruly that it is of use in spaces of low dimension only. Even sophisticated versions[5,6] developed for purely theoreti-

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cal reasons did not make it any more useful in applications. The second way is to use chains of reductive subgroups. In special cases[7] this produces perhaps the most explicit and desirable form of representation theory. Unfortunately, in most cases it offers at best only a simplification of the construction. The third method based on subalgebras and their centralizers is practically restricted to low rank groups[8] or to very particular classes of representations.[9] Systematic exploitation of CCO and exploitation of the finite group $N \subset K$ generated by them leads to a new approach to the problem, where the group N plays the role of the Weyl group of K "lifted" into the representation space.

Let us emphasize that we are concerned here with methods which apply to any simple (and by an obvious extension to any reductive) Lie group K and therefore we ignore existing vast literature applicable only to groups of particular type(s).

The most obvious obstacle to construction of basis for representation spaces for anything beyond those of very low dimension is the sheer enormity of the number of vectors to be written down. The natural way out of this is to compute only the dominant weight spaces (which in general make only a tiny fraction of the entire space) and to use the group N to move outside them when necessary for some problem at hand. This approach leads to two fundamental problems

(i) Build a "good" basis for each of the dominant weight spaces.

(ii) Describe how to move about in the rest of the representation space.

Consequently, the second objective of our article is to describe an approach to basis construction, at least as far as it is possible at this time, and to illustrate various aspects of it by numerous examples, because in many cases of practical interest it offers a considerable help already in its present form.

The problem (i) of building orthonormal bases in dominant weight spaces is the truly difficult part of the construction. Our examples in Sec. VI illustrate two approaches to solving it. The first uses tensor products of simple spaces (practical in many situations), the second one involves the representation theory of subgroups of N (eigenspace decomposition of stabilizers of dominant weight vectors). A third approach would be exploitation of various subgroups of K.

Whenever a particular (reductive) subgroup $K' \subset K$ is of importance in an application, it should be reflected in the basis construction. That is, the bases in dominant weight subspaces [problem (i) above] have to be built using N' of K' rather than N of K. Naturally, in the simple situation when the corresponding branching rule for $K' \subset K$ contains each K'- irreducible component at most once, there are the usual shortcuts so that one faces the same problem but for smaller representations of the smaller group K'.

Moving between weight spaces of the representation [problem (ii)] is accomplished by two processes: (a) moving along N-orbits of the space; (b) crossing orbits. The first process is carried out by the group N whose action is completely described in Sec. III. The second process is carried out by transforming dominant weight vectors of one space to another dominant weight subspace. This involves computing the action of a few generators between a few subspaces once for all. The Figs. 3 and 4 illustrate the succinct way in which this information can be presented.

In Sec. II we work out in detail and with only simple means the CCO in the SU(2) case. This serves as an introduction to the general situation described in Secs. III and IV, followed by some examples (Secs. V and VI).

II. THE CHARGE CONJUGATION OPERATOR OF SU(2)

In order to specify an irreducible representation of SU(2) of dimension $L+1$ we use (L), where L is related to the highest weight $\Lambda = j\beta$ of the representation by

$$L = 2(\Lambda,\beta)/(\beta,\beta) = 2j. \tag{2.1}$$

Hence j is the familiar "angular momentum." A convenient orthonormal basis consists of the vectors (angular momentum states) denoted by $|{}^{L}_{M}\rangle$, such that

$$M = 2(\mu,\beta)/(\beta,\beta),$$

where $\mu = M\beta/2$ is a weight of the weight system Ω_L of (L). Specifically,

$$M = 2(\Lambda - k\beta,\beta)/(\beta,\beta), \quad k \in \{0,1,..,L\}. \tag{2.2}$$

Thus

$$M \in \{L, L-2,..., -L\}. \tag{2.3}$$

Assuming $(\beta,\beta) = 2$, the action of the Lie algebra spanned by generators $e_\pm$ and h satisfying the commutation relations

$$[e_+, e_-] = h, \quad [h, e_\pm] = \pm 2e_\pm \tag{2.4}$$

on the basis vectors $|{}^{L}_{M}\rangle$ of the space V^L is given by

$$\begin{aligned} h\,|{}^{L}_{M}\rangle &= (\mu,\beta)|{}^{L}_{M}\rangle = M\,|{}^{L}_{M}\rangle, \\ e_\pm\,|{}^{L}_{M}\rangle &= \tfrac{1}{2}\sqrt{(L\mp M)(L\pm M+2)}\,|{}^{L}_{M\pm 2}\rangle. \end{aligned} \tag{2.5}$$

There is up to an inversion only one CCO in the rank one group SU(2). It is defined by

$$R = \exp e_- \,\exp(-e_+)\exp e_- \tag{2.6}$$

from which it follows that

$$R\,|{}^{L}_{M}\rangle = (-1)^{(L-M)/2}|{}^{L}_{\bar M}\rangle. \tag{2.7}$$

Let us illustrate (2.7) by an example:

$$\begin{aligned} R\,|{}^{3}_{1}\rangle &= \exp e_- \,\exp(-e_+)(1 + e_- + \tfrac{1}{2}e_-^2 + \cdots)\,|{}^{3}_{1}\rangle \\ &= \exp e_-(1 - e_+ + \tfrac{1}{2}e_+^2 - \tfrac{1}{6}e_+^3 + \cdots) \\ &\quad\times(|{}^{3}_{1}\rangle + 2|{}^{3}_{\bar 1}\rangle + \sqrt{3}\,|{}^{3}_{\bar 3}\rangle) \\ &= (1 + e_- + \tfrac{1}{2}e_-^2 + \tfrac{1}{6}e_-^3 + \cdots) \\ &\quad\times(\sqrt{3}|{}^{3}_{3}\rangle - |{}^{3}_{1}\rangle) = -|{}^{3}_{\bar 1}\rangle. \end{aligned} \tag{2.8}$$

Here and through the rest of the article we write $-a$ as $\bar a$ in matrixlike symbols. Repeated application of (2.7) gives

$$\begin{aligned} R^2|{}^{L}_{M}\rangle &= (-1)^L\,|{}^{L}_{M}\rangle = (-1)^M|{}^{L}_{M}\rangle, \\ R^4|{}^{L}_{M}\rangle &= |{}^{L}_{M}\rangle, \end{aligned} \tag{2.9}$$

which demonstrates that R is an element of order 4.

Note that CCO of SU(2) could have been defined by

$$\tilde R = \exp e_+ \,\exp(-e_-)\exp e_+, \tag{2.10}$$

which would imply the interchange of M and $-M$ in (2.7). Indeed one can verify directly that

$$\widetilde{R}\,|{}^{L}_{M}\rangle = (-1)^{(L+M)/2}|{}^{L}_{\bar{M}}\rangle\,. \tag{2.11}$$

Hence,

$$\widetilde{R} = R^{-1}\,. \tag{2.12}$$

The matrix of R relative to the basis $\{|{}^{L}_{M}\rangle\}$ for $L = 0,1,2,3,...$ is

$$R = (1), \quad R = \begin{pmatrix} 0 & \bar{1} \\ 1 & 0 \end{pmatrix},$$

$$R = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \bar{1} & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad R = \begin{pmatrix} 0 & 0 & 0 & \bar{1} \\ 0 & 0 & 1 & 0 \\ 0 & \bar{1} & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \tag{2.13}$$

and so on. The trace of any of the matrices R in (2.13) is the character of the element $R \in \mathrm{SU}(2)$ in the corresponding representation of SU(2). Thus tr R takes only three distinct values:

$$\operatorname{tr} R = \begin{cases} 1 & \text{for } L = 0 \quad (\mathrm{mod}\ 4) \\ 0 & \text{for } L = 1 \text{ or } 3 \quad (\mathrm{mod}\ 4)\,. \\ -1 & \text{for } L = 2 \quad (\mathrm{mod}\ 4) \end{cases} \tag{2.14}$$

It was shown in Ref. 10 that in SU(2) there is only one conjugacy class of elements of order 4 whose character values on irreducible representations are restricted to 0, ± 1,—Kostant's principal element—and, in notations of Kac (cf. Refs. 10, 11, and 12), its conjugacy class is given as $R \sim [11]$.

Subsequently we need the transformation of generators $e_{\pm}$ and h by R. For that consider the equalities

$$\begin{aligned}
|{}^{1}_{1}\rangle &= h\,|{}^{1}_{1}\rangle = -R\,|{}^{1}_{\bar{1}}\rangle \\
&= Rh\,|{}^{1}_{\bar{1}}\rangle = RhR^{-1}R\,|{}^{1}_{\bar{1}}\rangle = -RhR^{-1}|{}^{1}_{1}\rangle\,, \\
|{}^{1}_{1}\rangle &= e_{+}\,|{}^{1}_{\bar{1}}\rangle = -R\,|{}^{1}_{\bar{1}}\rangle = -Re_{-}|{}^{1}_{1}\rangle \\
&= -Re_{-}R^{-1}R\,|{}^{1}_{1}\rangle = -Re_{-}R^{-1}|{}^{1}_{\bar{1}}\rangle\,, \\
|{}^{1}_{\bar{1}}\rangle &= e_{-}|{}^{1}_{1}\rangle = R\,|{}^{1}_{1}\rangle = Re_{+}\,|{}^{1}_{\bar{1}}\rangle \\
&= Re_{+}R^{-1}R\,|{}^{1}_{\bar{1}}\rangle = -Re_{+}R^{-1}|{}^{1}_{1}\rangle\,,
\end{aligned} \tag{2.15}$$

from which it follows immediately that

$$RhR^{-1} = -h, \quad Re_{\pm}R^{-1} = -e_{\mp}\,. \tag{2.16}$$

Finally, let us also point out that

$$R\,|{}^{L}_{0}\rangle = R^{-1}|{}^{L}_{0}\rangle = (-1)^{L/2}|{}^{L}_{0}\rangle\,. \tag{2.17}$$

III. CHARGE CONJUGATION OPERATORS OF ARBITRARY SIMPLE COMPACT LIE GROUP

All ideas of this section extend naturally to arbitrary simply connected compact Lie groups, but for simplicity we consider here only simple simply connected compact Lie groups. The purpose of this section is to bring together known facts relevant to CCO.

Let $\mathbf{k}$ be the Lie algebra of a simple simply connected compact Lie group K, $\mathbf{g}$ its complexification $\mathbf{k}_{\mathrm{C}}$, and G the simply connected complex group with Lie algebra $\mathbf{g}$ and with maximal compact subgroup K. We let T be a maximal torus of G and $\mathbf{h}$ be the corresponding Cartan subalgebra of $\mathbf{g}$. Thus $\mathbf{h} = \mathbf{t}_{\mathrm{C}}$, where $\mathbf{t}$ is the subalgebra of $\mathbf{g}$ corresponding to T, and $\mathbf{h}_{\mathbf{R}} := \sqrt{-1}\ \mathbf{t}$ is a real Euclidean space (under the Killing form) of dimension $l = \mathrm{rank}(G)$. Here the symbol $:=$ indicates that the left side is defined by the right one.

Relative to $\mathbf{h}$ we have the root space decomposition

$$\mathbf{g} = \mathbf{h} \oplus \bigoplus_{\alpha \in \Delta} \mathbf{g}^{\alpha} \tag{3.1}$$

of $\mathbf{g}$, where $\Delta \subset \mathbf{h}^{*}_{\mathbf{R}}$ (the dual space to $\mathbf{h}_{\mathbf{R}}$) is the root system of $\mathbf{g}$ relative to $\mathbf{h}$, and for each $\alpha \in \Delta$,

$$\mathbf{g}^{\alpha} = \{x \in g\,|\,[h,x] = \alpha(h)x \quad \text{for all } h \in \mathbf{h}\}. \tag{3.2}$$

Choosing an ordering of $\mathbf{h}^{*}_{\mathbf{R}}$ leads to an ordering on Δ. Let Δ^{+} denote the corresponding set of positive roots in Δ and let $\Pi = \{\alpha_1,...,\alpha_l\}$ be the corresponding set of simple roots.

For each $\beta \in \Delta$, $\mathbf{sl}^{\beta}(2) = \mathbf{g}^{\beta} + \mathbf{g}^{-\beta} + [\mathbf{g}^{\beta},\mathbf{g}^{-\beta}]$ is a subalgebra of $\mathbf{g}$ isomorphic to $\mathbf{sl}(2)$, and we chose $e_{\beta} \in g^{\beta}$, $e_{-\beta} \in \mathbf{g}^{-\beta}$, and $h_{\beta} \in \mathbf{h}$ such that

$$[h_{\beta},e_{\pm\beta}] = \pm 2\, e_{\pm\beta}, \quad [e_{\beta},e_{-\beta}] = h_{\beta}\,. \tag{3.3}$$

These h_{β} are uniquely determined by (3.3) and satisfy

$$\lambda(h_{\beta}) = 2(\lambda,\beta)/(\beta,\beta) \tag{3.4}$$

for all $\lambda \in \bar{h}^{*}$. The choice of $e_{\beta} \in \mathbf{g}^{\beta}$, $e_{\beta} \neq 0$, is arbitrary, whereupon $e_{-\beta}$ is uniquely determined by $[e_{\beta},e_{-\beta}] = h_{\beta}$. At this time we leave this choice free.

Let $G^{\beta} \subset G$ be the connected subgroup whose Lie algebra is $\mathbf{sl}^{\beta}(2)$, and let $\mathrm{SU}^{\beta}(2) = G^{\beta} \cap K = \langle \exp(\mathbf{sl}^{\beta}(2) \cap \mathbf{k})\rangle$ be the corresponding compact subgroup. Thus $G^{\beta} \cong \mathrm{SL}_{\mathrm{C}}(2)$ and $K^{\beta} \cong \mathrm{SU}(2)$.

A specific isomorphism of G^{β} and $\mathrm{SL}_{\mathrm{C}}(2)$ is established by identifying $\mathbf{sl}^{\beta}(2)$ and sl(2):

$$e_{\beta} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad e_{-\beta} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h_{\beta} = \begin{pmatrix} 1 & 0 \\ 0 & \bar{1} \end{pmatrix}. \tag{3.5}$$

Consequently,

$$\exp(-e_{\beta}) = \begin{pmatrix} 1 & \bar{1} \\ 0 & 1 \end{pmatrix}, \quad \exp e_{-\beta} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \tag{3.6}$$

and

$$R_{\beta} := \exp e_{-\beta} \exp(-e_{\beta}) \exp e_{-\beta} = \begin{pmatrix} 1 & \bar{1} \\ 1 & 0 \end{pmatrix} \in \mathrm{SU}(2) \tag{3.7}$$

as in (2.13).

Now let $\rho: K \to \mathrm{GL}(V)$ be a finite-dimensional (unitary) representation of K on a complex space V and let $d\rho: \mathbf{k} \to \mathrm{End}(V)$ be its differential, i.e., a representation of $\mathbf{k}$ on V. Both ρ and $d\rho$ have complexifications,

$$\rho_{\mathrm{C}}: G \to \mathrm{GL}(V),$$

$$d\rho_{\mathrm{C}}: \mathbf{g} \to \mathrm{End}(V).$$

Relative to T, V decomposes into weight spaces

$$V = \bigoplus_{\lambda \in \Omega} V(\lambda)\,, \tag{3.8}$$

where $\Omega \subset \mathbf{h}^{*}_{\mathbf{R}}$ is the weight system and for all $\lambda \in \Omega$

$$V(\lambda) = \{v \in V\,|\,d\rho_{\mathrm{C}}(h)\,v = \lambda(h)v \quad \text{for all } h \in \mathbf{h}\}\,. \tag{3.9}$$

For each $\beta \in \Delta$ we have, by restriction, representations

$$\rho^{\beta}: \mathrm{SU}^{\beta}(2) \to \mathrm{GL}(V),$$

$$\rho_{\mathrm{C}}^{\beta}: G^{\beta} \to \mathrm{GL}(V),$$
$$d\rho_{\mathrm{C}}^{\beta}: \mathbf{sl}_{\beta}(2) \to \mathrm{End}(V).$$

If we partition Ω into β-weight strings, $(\lambda + \mathbf{Z}\beta) \cup \Omega$, i.e.,

$$\lambda + q\beta, \lambda + (q-1)\beta, \ldots, \lambda, \ldots, \lambda - p\beta, \tag{3.10}$$

then the sums of the corresponding weight spaces

$$\bigoplus_{j=-p}^{q} V(\lambda + j\beta) \tag{3.11}$$

are $\mathrm{SU}^{\beta}(2)$-submodules. They can further be decomposed into $\mathrm{SU}^{\beta}(2)$-irreducible submodules. Each such submodule is a sum

$$\bigoplus_{k=0}^{s} \mathbf{C}v(\Lambda - k\beta), \tag{3.12}$$

where $\Lambda = \lambda + r\beta$ for some r and $v(\Lambda - k\beta) \in V(\Lambda - k\beta)$. The identification (3.5) of $\mathbf{sl}^{\beta}(2)$ and sl(2) leads to representations of $\mathbf{sl}(2)$ and $\mathrm{SL}_{\mathrm{C}}(2)$ on V such that

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \to d\rho_{\mathrm{C}}(e_{\beta}),$$
$$\begin{pmatrix} 1 & \bar{1} \\ 0 & 1 \end{pmatrix} \to \exp(d\rho_{\mathrm{C}}(-e_{\beta})) = \rho_{\mathrm{C}}(\exp(-e_{\beta})), \tag{3.13}$$
$$R_{\beta} \to \rho_{\mathrm{C}}(\exp e_{-\beta} \exp(-e_{\beta}) \exp e_{-\beta}), \text{ etc.}$$

An appropriate choice of $v(\Lambda - k\beta)$ allows us to identify them with $|{}^{L}_{M}\rangle$ of Sec. II, namely,

$$v(\Lambda - k\beta) \leftrightarrow \left|{}^{\Lambda(h_{\beta})}_{(\Lambda - k\beta)(h_{\beta})}\right\rangle = |{}^{L}_{M}\rangle. \tag{3.14}$$

Then also $e_{\pm\beta}$ and h_{β} act on (3.14) according to (2.5).

Although R_{β} is defined in terms of nonunitary operators, it lies in $\mathrm{SU}(2) \subset K$ and hence appears as a unitary operator on V. From now on we write R_{β} for $\rho_{\mathrm{C}}(R_{\beta})$. According to (2.7), one has

$$R_{\beta}\left|{}^{\Lambda(h_{\beta})}_{(\Lambda - k\beta)(h_{\beta})}\right\rangle = (-1)^{k}\left|{}^{\Lambda(h_{\beta})}_{(\Lambda - k\beta)(-h_{\beta})}\right\rangle, \tag{3.15}$$

which demonstrates the "charge conjugating role" of the operators R_{β}. Thus the general effect of R_{β} on V is the permutation of weight subspaces:

$$V(\lambda) \leftrightarrow V(\lambda - \lambda(h_{\beta})\beta). \tag{3.16}$$

The Weyl reflection $r_{\beta}: \mathbf{h}_{\mathbf{R}}^{*} \to \mathbf{h}_{\mathbf{R}}^{*}$ is defined by

$$r_{\beta}\lambda = \lambda - \lambda(h_{\beta})\beta = \lambda - (2(\lambda, \beta)/(\beta, \beta)\beta). \tag{3.17}$$

Therefore,

$$R_{\beta}V(\lambda) = V(r_{\beta}\lambda). \tag{3.18}$$

The Weyl group is by definition the group W generated by the $r_{\alpha}, \alpha \in \Delta$. For all $\alpha, \beta \in \Delta$, $r_{\alpha}^{2} = 1$, and $r_{\alpha}r_{\beta}r_{\alpha} = r_{r_{\alpha}\beta}$. It follows that W is generated by the $r_{i} := r_{\alpha_{i}}$, $i = 1,2,\ldots,l$.

Whereas, $r_{\beta}^{2} = 1$, it is obvious from (2.9) that we only have

$$R_{\beta}^{4} = 1. \tag{3.19}$$

From (2.9) and (3.15) one has

$$R_{\beta}^{2}|V(\lambda) = (-1)^{\lambda(h_{\beta})}. \tag{3.20}$$

Following Tits[13] we write

$$R_{\beta}^{2} = (-1)^{h_{\beta}}. \tag{3.21}$$

Corresponding to the generators $e_{\beta}, e_{-\beta}, h_{\beta}$ of $\mathbf{sl}^{\beta}(2)$, we have

$$e_{-\beta}, \quad e_{\beta}, \quad h_{-\beta} = -h_{\beta} \tag{3.22}$$

as a set of generators of $\mathbf{sl}^{\beta}(2)$. The operator $R_{-\beta}$ is then

$$R_{-\beta} = \exp e_{\beta} \exp(-e_{-\beta}) \exp e_{\beta} \leftrightarrow \begin{pmatrix} 0 & 1 \\ \bar{1} & 0 \end{pmatrix} \leftrightarrow R_{\beta}^{-1}.$$

Thus

$$R_{-\beta} = R_{\beta}^{-1}. \tag{3.23}$$

IV. THE FINITE GROUP GENERATED BY CHARGE CONJUGATION OPERATORS

Our primary interest here is the group N generated by $R_{\beta}, \beta \in \Delta$. As it stands N depends on the choice of e_{β} (hence, R_{β}), $\beta \in \Delta$. The most convenient form of N arises by the use of a Chevalley basis[5,6] of $\mathbf{g}$. According to Ref. 14, there is a choice of the $e_{\beta}, \beta \in \Delta$, such that the following occurs: For $\alpha, \beta \in \Delta$, where α and β are linearly independent with root string $\beta - p\alpha, \ldots, \beta, \ldots, \beta + q\alpha$,

$$[e_{\alpha}, e_{\beta}] = \pm(p+1)e_{\alpha+\beta} \text{ if } \alpha + \beta = \Delta. \tag{4.1}$$

The matter of sign is not essential to us here. With such a choice of basis,

$$\mathbf{g}_{\mathbf{Z}} := \sum \mathbf{Z}h_{i} + \sum_{\beta \in \Delta} \mathbf{Z}e_{\beta} \tag{4.2}$$

is a Lie ring. Most importantly, for all $\beta \in \Delta$ and for all $n \in \mathbf{N}$,

$$(1/n!)(ad e_{\beta})^{n}: \mathbf{g}_{\mathbf{Z}} \to \mathbf{g}_{\mathbf{Z}}. \tag{4.3}$$

More generally, Kostant has shown[6,15] that for every representation (ρ, V) of $\mathbf{g}$ there is a basis $v_{1}, \ldots, v_{m}$ of V consisting of weight vectors such that if we set

$$V_{\mathbf{Z}} = \bigoplus_{j} \mathbf{Z}v_{j}, \tag{4.4}$$

then for all $\beta \in \Delta$

$$\frac{1}{n!}(d\rho(e_{\beta}))^{n}: V_{\mathbf{Z}} \to V_{\mathbf{Z}}. \tag{4.5}$$

Here $\{e_{\beta}\}$ are assumed to be a Chevalley basis. Thus in particular,

$$R_{\beta}: V_{\mathbf{Z}} \to V_{\mathbf{Z}}. \tag{4.6}$$

Since $R_{-\beta} = R_{\beta}^{-1}$ we see that R_{β} is a bijective mapping of $V_{\mathbf{Z}}$.

From the Chevalley basis the operators R_{α} and R_{α}^{-1} map $\mathbf{g}_{\mathbf{z}}$ into itself (in the adjoint representation) and are automorphisms of $\mathbf{g}_{\mathbf{z}}$ as a Lie ring. It is easy to check [cf. (2.16)] that $R_{\alpha}h_{\alpha} = -h_{\alpha}$ and R_{α} acts trivially on the orthogonal complement of h_{α} in $\mathbf{h}_{\mathbf{R}}$. Thus

$$R_{\alpha}h_{\beta} = h_{r_{\alpha}\beta}. \tag{4.7}$$

Also since for each $\gamma \in \Delta$, the generators $\pm e_{\gamma}$ are the only ones for $\mathbf{g}_{\mathbf{Z}}^{\alpha} = \mathbf{Z}e_{\gamma}$ (as $\mathbf{Z}$-modules),

$$R_{\alpha}e_{\beta} = \pm e_{r_{\alpha}\beta}, \quad \text{for all } \alpha, \beta \in \Delta. \tag{4.8}$$

From $R_{\alpha}(h_{\beta}) = R_{\alpha}[e_{\beta}, e_{-\beta}] = [R_{\alpha}e_{\beta}, R_{\alpha}e_{-\beta}]$ it follows that

$$R_{\alpha}e_{-\beta} = \pm e_{-r_{\alpha}\beta}, \tag{4.9}$$

where the sign is the same as in (4.8). From this we have

$$R_\alpha R_\beta R_\alpha^{-1} = R_\alpha \exp(-e_\beta)\exp e_{-\beta}\exp(-e_\beta)R_\alpha^{-1}$$
$$= \exp(\mp e_{r_\alpha \beta})\exp(\pm e_{-r_\alpha\beta})\exp(\mp e_{r_\alpha\beta})$$
$$= R_{r_\alpha \beta}^{\pm 1}. \qquad (4.10)$$

It follows that the group N is generated by

$$R_i := R_{\alpha_i}, \quad i = 1,2,\dots,l. \qquad (4.11)$$

A direct consequence of the well-known defining identities $(r_i r_j)^k = 1$ of the Weyl group are the identities

$$\underbrace{R_i R_j R_i R_j \cdots}_{k \text{ factors}} = \underbrace{R_j R_i R_j R_i \cdots}_{k \text{ factors}}. \qquad (4.12)$$

Next consider the group

$$A := \langle R_1^2,\dots,R_l^2\rangle \qquad (4.13)$$

generated by R_i^2. According to (3.21), R_i^2 commute and the general element

$$R_1^{2\epsilon_1}\cdots R_l^{2\epsilon_l}, \quad \epsilon^i = 0 \text{ or } 1,\ i = 1,2,\dots,l,$$

of A acts as $(-1)^{\epsilon_1 h_1 + \cdots + \epsilon_l h_l}$.

The abelian group A can thus be identified as

$$\mathbf{h}_{\mathbf{Z}_2} := (\mathbf{Z}h_1 + \cdots + \mathbf{Z}h_l)/(2\mathbf{Z}h_1 + \cdots + 2\mathbf{Z}h_l)$$
$$\cong (\mathbf{Z}/2\mathbf{Z})\times\cdots\times(\mathbf{Z}/2\mathbf{Z}) \quad (l \text{ factors}) \qquad (4.14)$$

and with $\alpha \in A$ corresponding to $\widetilde{\alpha} \in \mathbf{h}_{\mathbf{Z}_2}$ and acting as

$$(-1)^{\widetilde{\alpha}} : (-1)^{\widetilde{\alpha}}|_{V(\lambda)} = (-1)^{\lambda(\alpha)}. \qquad (4.15)$$

It is convenient to write $\lambda(\widetilde{\alpha})$ for $\lambda(\alpha)$ mod 2 so that $(-1)^{\lambda(\widetilde{\alpha})} = (-1)^{\lambda(\alpha)}$. In view of (3.18) there is a natural mapping

$$\pi: N \to W \qquad (4.16)$$

with $R_\alpha \mapsto r_\alpha$ and

$$RV(\lambda) = V(w\lambda) \qquad (4.17)$$

for all weight spaces $V(\lambda)$ when $\pi(R) = w$. Clearly A is in the kernel of π. In fact A is the kernel of π and we have the exact sequence

$$1 \to A \to N \xrightarrow{\pi} W \to 1. \qquad (4.18)$$

We can see as follows that (4.18) holds. Suppose that $R \in N$ and $\pi(R) = 1$. Then R stabilizes each weight space in every irreducible representation of K. In particular, $Ad(R)e_\alpha = \pm e_\alpha$ for each $\alpha \in \Delta$ so that $(AdR)^2 = 1$ and R^2 is in the center of K. Thus R^2 acts as a scalar on each irreducible representation (ρ,V). However, R stabilizes the top weight space (one dimensional) and using Kostant basis[6] we again see $R^2 = 1$. Now $R = \exp 2\pi i h$, for $h \in \mathbf{h}_\mathbf{R}$ and $R^2 = 1$, implies that $h = \frac{1}{2}(\epsilon_1 h_1, + \cdots + \epsilon_l h_l)$, $\epsilon_i = 0,1$. Thus $R = (-1)^{\overline{\alpha}} \in A$, where $\overline{\alpha}$ is $\epsilon_1 h_1 + \cdots + \epsilon_l h_l$ taken in $\mathbf{h}_{\mathbf{Z}_2}$.

From (4.18) we have for the order $|N|$ of N:

$$|N| = 2^l |W|. \qquad (4.19)$$

Since A is a abelian and $A \lhd N$, the action $\alpha \mapsto nan^{-1}$ of N on A determines an action of W on $\mathbf{h}_\mathbf{R}$. This is easy to specify. Recall that N acts on $\mathbf{h}_\mathbf{R}$ by (4.7). Since $R_\beta^2 \mathbf{h}_\mathbf{R} = 1$ for all $\beta \in \Delta$, A acts trivially on $\mathbf{h}_\mathbf{R}$ and we have a representation of W on $\mathbf{h}_\mathbf{R}$. Precisely,

$$r_\alpha : h \to h - \alpha(h)h_\alpha. \qquad (4.20)$$

This is no more than the action of W on $\mathbf{h}_\mathbf{R}$ induced by transposing the action of W on $\mathbf{h}_{\mathbf{R}^*}$: For all $w \in \mathbf{h}_{\mathbf{R}^*}$, $h \in \mathbf{h}_\mathbf{R}$,

$$w\varphi(h) = \varphi(w^{-1}h). \qquad (4.21)$$

Since W stabilizes $\mathbf{h}_\mathbf{Z} = \Sigma\, \mathbf{Z}h_i$, it thus produces a modulo 2 action on $\mathbf{h}_{\mathbf{Z}_2}$. Let $\overline{\alpha} \in \mathbf{h}_{\mathbf{Z}_2}$ and $(-1)^{\overline{\alpha}}$ be the corresponding element of A. Then

$$w\cdot(-1)^{\overline{\alpha}} = (-1)^{w\overline{\alpha}}. \qquad (4.22)$$

If (ρ,V) is any unitary representation of K and $V = \oplus_{\lambda\in\Omega} V(\lambda)$ is the weight space decomposition of V, then we have seen

$$RV(\lambda) = V(\pi(R)\lambda)$$

and $(-1)^{\overline{\alpha}}|_{V(\lambda)} = (-1)^{\lambda(\overline{\alpha})}$ [cf. (4.18) and (4.15)]. Important subgroups of N are those which stabilize a given weight space

$$N_\lambda = \{R \in N \,|\, RV(\lambda) = V(\lambda)\}. \qquad (4.23)$$

As suggested by the notation, N_λ does in fact only depend on λ (not on the representation). An explicit description of N_λ is given by the following considerations. Each W-orbit, $W\lambda$ ($\lambda \in \Omega$), contains unique dominant element λ^+: defined by

$$\lambda^+(h_i) \geqslant 0 \quad \text{for all } i = 1,\dots,l. \qquad (4.24)$$

For λ^+ dominant, let

$$J := \{i \in \{1,\dots,l\} \,|\, \lambda^+(h_i) = 0\}. \qquad (4.25)$$

Then N_{λ^+} is the group generated by A and by the R_i, $i \in J$. Alternatively, if we define

$$W_J := \langle r_i \,|\, i \in J\rangle, \qquad (4.26)$$

then N_{λ^+} is the full preimage $N_J := \pi^{-1}(W_J)$ of W_J in N,

$$1 \to A \to N_J \to W_J \to 1. \qquad (4.27)$$

The cardinality of the set $W\lambda^+ = \{w\lambda^+ \,|\, w \in W\}$ is precisely the index

$$[W:W_J] = |W|/|W_J| \qquad (4.28)$$

of W_J in W. This is trivial to compute since W_J is the Weyl group of the subroot system of Δ based on $\{\alpha_j \,|\, j \in J\}$. For λ not dominant, choose $w \in W$ such that $\lambda^+ = w^{-1}\lambda$ is dominant, and define N_{λ^+} as above. Then

$$N_\lambda = wN_{\lambda^+}w^{-1}. \qquad (4.29)$$

Note that (4.29) makes sense since the choice of representation R of w in N for computing (4.29) is immaterial.

V. EXAMPLES

In order to illustrate the content of Sec. III and IV, we consider here some particular cases.

A. The group SU(3)

Consider the lowest faithful representation with the highest weight $\Lambda = (1,0)$ of the group SU(3). Its representation space V^Λ is spanned by the weight vectors

$$\left|{}^{10}_{10}\right\rangle, \quad \left|{}^{10}_{\bar{1}1}\right\rangle, \quad \text{and } \left|{}^{10}_{0\bar{1}}\right\rangle. \qquad (5.1)$$

For simplicity we omit the highest weight in symbols like (5.1) whenever there can be no ambiguity. According to (3.15).

$$R_1|10\rangle = |\bar{1}1\rangle, \quad R_1|\bar{1}1\rangle = -|10\rangle, \quad R_1|0\bar{1}\rangle = |0\bar{1}\rangle,$$
$$R_2|10\rangle = |10\rangle, \quad R_2|\bar{1}1\rangle = |0\bar{1}\rangle, \quad R_2|0\bar{1}\rangle = -|\bar{1}1\rangle, \tag{5.2}$$

and therefore we have in the SU(3)-representation (1,0)

$$R_1 = \begin{pmatrix} 0 & \bar{1} & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \bar{1} \\ 0 & 1 & 0 \end{pmatrix},$$
$$R_1R_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \tag{5.3}$$
$$R_1^2 = \begin{pmatrix} \bar{1} & 0 & 0 \\ 0 & \bar{1} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad R_2^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \bar{1} & 0 \\ 0 & 0 & \bar{1} \end{pmatrix}, \text{ etc.}$$

By a direct computation one is led to the conclusion that the subgroup $N \subset \mathrm{SU}(3)$ generated by R_1 and R_2 is isomorphic to the octahedral group O of order 24 and that the above representation is the three-dimensional irreducible representation with determinant of all elements equal one. Adopting notations of Ref. 16, it is the representation Γ_4.

Since $N \subset \mathrm{SU}(3)$, every SU(3) representation (p,q) reduces with respect to N. That is

$$(p,q) \supset \bigoplus_{i=1}^{5} m_i \Gamma_i, \tag{5.4}$$

where m_i is the multiplicity of Γ_i in the reduction. The multiplicities are easily found for any (p,q) from the generating function (4.6)–(4.10) of Ref. 17.

The elements R_1 and R_2 of N lie in the same SU(3)-conjugacy class of regular elements of order 4 in SU(3), namely the one denoted by [2 1 1] in Table I of Ref. 10. Their character values on irreducible SU(3)-representations are restricted to 0, ± 1.

B. The group SU(n)

As in the previous case one finds a faithful matrix representation of $N \subset \mathrm{SU}(n)$ by considering the action of R_i, $i = 1,2,\ldots,n-1$, in the lowest faithful representation $(1,0,\ldots,0)$ of SU(n) according to (3.15):

$$R_i = I_{i-1} \oplus \begin{pmatrix} 0 & \bar{1} \\ 1 & 0 \end{pmatrix} \oplus I_{n-i-1}, \quad i = 1,2,\ldots,n-1, \tag{5.5}$$

where I_k is the $k \times k$ identity matrix. From (4.19) we find that the order of N is $|N| = 2^{n-1}n!$

It is obvious in (5.5) that all R_i belong to the same SU(n)-conjugacy class of rational elements of order 4, which is identified as [210...01] in Table 6 of Ref. 12. Except for SU(2) and SU(3), the R_i are not regular in SU(n) and consequently the set of their character values over all irreducible SU(n) representations is an unbounded set of integers. The elements R_i satisfy the following identities

$$\begin{aligned} &R_i^4 = 1, \\ &R_iR_j = R_jR_i, \quad \text{if } |i-j| > 1. \\ &R_iR_jR_i = R_jR_iR_j \quad \text{if } |i-j| = 1. \end{aligned} \tag{5.6}$$

The exact sequence (4.18) can be written as

$$1 \to \mathbf{Z}_2 \times \underset{n-1 \text{ times}}{\cdots} \times \mathbf{Z}_2 \to N \to S_n \to 1\,. \tag{5.7}$$

Here the Weyl group SU(n) is isomorphic to the symmetric group S_n of n letters.

C. The groups USp(4) and O(5)

In the symplectic four-dimensional representation (1,0) of these groups, we have

$$R_1 = \begin{pmatrix} 0 & \bar{1} & & \\ 1 & 0 & & \\ & & 0 & \bar{1} \\ & & 1 & 0 \end{pmatrix} \text{ and } R_2 = \begin{pmatrix} 1 & & & \\ & 0 & \bar{1} & \\ & 1 & 0 & \\ & & & 1 \end{pmatrix} \tag{5.8}$$

relative to the basis of weight vectors. Therefore, also

$$\begin{aligned} &R_1^4 = R_2^4 = 1, \\ &R_1R_2R_1R_2 = R_2R_1R_2R_1\,. \end{aligned} \tag{5.9}$$

Similarly in the five-dimensional orthogonal representation (0,1), one has

$$R_1 = \begin{pmatrix} 1 & & & & \\ & 0 & 0 & 1 & \\ & 0 & \bar{1} & 0 & \\ & 1 & 0 & 0 & \\ & & & & 1 \end{pmatrix} \text{ and } R_2 = \begin{pmatrix} 0 & \bar{1} & & & \\ 1 & 0 & & & \\ & & 1 & & \\ & & & 0 & \bar{1} \\ & & & 1 & 0 \end{pmatrix}, \tag{5.10}$$

which also satisfy the identities (5.9). The exact sequence (4.18) is in this case

$$1 \to \mathbf{Z}_2 \times \mathbf{Z}_2 \to N \to D_4 \to 1, \tag{5.11}$$

where D_4 is the dihedral group. The order of N is 32. The elements R_1 and R_2 are not conjugate to each other because they correspond to simple roots of different length. Their conjugacy classes are identified in Table 6 of Ref. 12 as [201] and [210], respectively, for R_1 and R_2. Both elements are rational, which implies that their characters take only integer values in any representation of the Lie group, but they are not regular which means that their character values are unlimited. For any given representation (a,b) the characters R_1 and R_2 are easily found from the generating function of Table V of Ref. 10.

D. The group G_2

In the lowest representation (0,1) of dim = 7 of G_2, one has R_1 and R_2 as

$$R_1 = 1 \oplus \begin{pmatrix} 0 & \bar{1} \\ 1 & 0 \end{pmatrix} \oplus 1 \oplus \begin{pmatrix} 0 & \bar{1} \\ 1 & 0 \end{pmatrix} \oplus 1,$$
$$R_2 = \begin{pmatrix} 0 & \bar{1} \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & 1 \\ 0 & \bar{1} & 0 \\ 1 & 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & \bar{1} \\ 1 & 0 \end{pmatrix}, \tag{5.12}$$

relative to the weight vector basis. The group N generated by (5.12) is of order 48. One has the exact sequence

$$1 \to \mathbf{Z}_2 \times \mathbf{Z}_2 \to N \to D_6 \to 1, \tag{5.13}$$

where D_6 is the dihedral group, and the identities

$$R_1^4 = R_2^4 = 1,$$
$$R_1R_2R_1R_2R_1R_2 = R_2R_1R_2R_1R_2R_1 . \tag{5.14}$$

The elements R_1 and R_2 are rational and nonregular in G_2. Their G_2-conjugacy classes are, respectively [201] and [110]. Their characters in any representation of G_2 are found from the generating functions of Table VI of Ref. 10.

VI. CONSTRUCTING BASES IN REPRESENTATION SPACES

In this section we demonstrate and illustrate the reduction of the problem of constructing a basis and computing the matrix elements of generators to similar problems of much smaller size involving only the dominant weight vectors.

Given the general decomposition (3.8) of a space V in which a compact simple Lie group K acts irreducibly as a representation ρ, it is natural to consider the present problem as a construction of bases in every weight subspace $V(\lambda) \subset V$, where $\lambda \in \Omega$. Let us point out that in almost every application, and certainly in all of them in elementary particle physics, one chooses a basis of weight vectors relative to some Cartan subalgebra of $\mathbf{g}$ whenever an explicit use of a basis is made. Otherwise one could not associate quantum numbers with the basis vectors–physical states. The dimension of $V(\lambda)$ is the multiplicity of λ in Ω so that the basis construction in $V(\lambda)$ is a nontrivial problem only when $\dim V(\lambda) > 1$.

Using notations of Sec. III, let us recall[14] the following. For every weight $\lambda \in \Omega$ there exists a unique dominant weight $\lambda^+ \in \Omega$ such that $\lambda \in W\lambda^+$. If $\lambda = w\lambda^+$ then also $\lambda = w\,\mathrm{Stab}_W(\lambda^+)(\lambda^+)$. There exists a unique canonical w such that

$$\lambda = w\lambda^+, \quad w = r_i r_{i_2} \cdots r_{i_k} , \tag{6.1}$$

in which the number k of reflections r_i is minimal. We define

$$\tilde{w} = R_{i_1} R_{i_2} \cdots R_{i_k} . \tag{6.2}$$

Although w is unique, its expression as a word in the reflections $r_1,\ldots,r_l$ is not. Thus $\tilde{w}$ depends on the choice of the writing w in (6.1). If $r_{j_1}\cdots r_{j_m}$ is any other expression for w (minimal or not), then $\tilde{w}' = R_{j_1}\cdots R_{j_m}$ is some other preimage of w in N and $\tilde{s} := \tilde{w}^{-1}\tilde{w}'$ stabilizes the weight space $V(\lambda^+)$. Hence it is unavoidable to consider the effect of such elements $\tilde{s} \in N$ on the weight spaces they stabilize.

Let us assume that for each dominant weight $\lambda^+ \in \Omega$ we have an orthogonal basis [problem (i) of Introduction]

$$|\lambda^+\rangle_1,\ldots,|\lambda^+\rangle_m \quad \text{of} \quad V(\lambda^+), \dim V(\lambda^+) = m . \tag{6.3}$$

Then for λ as in (6.1) we define

$$\tilde{w}|\lambda^+\rangle_1, \tilde{w}|\lambda^+\rangle_2,\ldots,\tilde{w}|\lambda^+\rangle_m \tag{6.4}$$

as our basis for $V(\lambda)$, which solves part (a) of problem (ii) of the Introduction. In the following five examples we illustrate our approach to that problem.

Example 1 [SU(2)]: All $V(\lambda)$ are one dimensional. A basis of $V(\lambda^+)$ consists of $|{}^L_M\rangle$, $M \geqslant 0$. As a basis vector in $V(-\lambda^+)$ we take

$$|{}^L_{\bar{M}}\rangle = (-1)^{(L-M)/2} R\, |{}^L_M\rangle , \tag{6.5}$$

where we have used the phase factor in order to keep the convention in complete agreement with Sec. II.

Example 2 [SU(3) representation (1,0)]: There is only one dominant weight $\lambda^+ = (1,0)$ in Ω of multiplicity 1. The basis is given in (5.2).

Example 3 [the adjoint representation (2,0) of Sp(4)]: By assumption we know the basis vectors

$$|20\rangle, \quad |01\rangle, \quad |00\rangle_1, \quad |00\rangle_2.$$

Then the rest is given by

$$\begin{aligned} &R_1|20\rangle = |\bar{2}2\rangle, && R_2|01\rangle = |2\bar{1}\rangle,\\ &R_2R_1|20\rangle = |2\bar{2}\rangle, && R_1R_2|01\rangle = |\bar{2}1\rangle,\\ &R_1R_2R_1|20\rangle = |\bar{2}0\rangle, && R_2R_1R_2|01\rangle = |0\bar{1}\rangle . \end{aligned} \tag{6.6}$$

Example 4 [SU(3) representation (3,2) of dim = 42]: Properties of dominant weights are shown on Fig. 1. Assuming that we have pairwise orthogonal dominant weight vectors

$$\begin{aligned} &|21\rangle_i, \quad |02\rangle_i, \quad i = 1,2,\\ &|10\rangle_j, \quad j = 1,2,3, \end{aligned} \tag{6.7}$$

the basis is given by

$$\begin{aligned} &|32\rangle, && R_2|\bar{3}1\rangle = R_2R_1|32\rangle = |\bar{2}\bar{1}\rangle,\\ &R_1|32\rangle = |\bar{3}1\rangle, && R_1|5\bar{2}\rangle = R_1R_2|32\rangle = |\bar{5}3\rangle,\\ &R_2|32\rangle = |5\bar{2}\rangle, && R_2|\bar{5}3\rangle = R_2R_1R_2|32\rangle = |\bar{2}\bar{3}\rangle,\\ &|13\rangle, && R_2|\bar{1}4\rangle = R_2R_1|13\rangle = |3\bar{4}\rangle,\\ &R_2|13\rangle = |2\bar{3}\rangle, && R_1|3\bar{4}\rangle = R_1R_2R_1|13\rangle = |\bar{3}\bar{1}\rangle,\\ &|40\rangle, && \\ &R_1|40\rangle = |\bar{4}4\rangle, && R_2|\bar{4}4\rangle = R_2R_1|40\rangle = |0\bar{4}\rangle,\\ &|02\rangle_i, i = 1,2, && R_1|2\bar{2}\rangle_i = R_1R_2|02\rangle_i = |\bar{2}0\rangle_i,\\ &R_2|02\rangle_i = |2\bar{2}\rangle_i ' && \end{aligned}$$

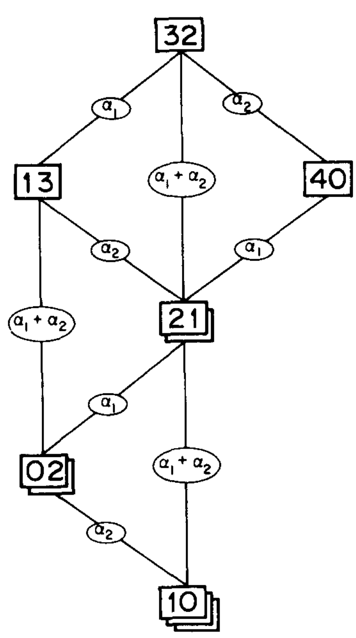

FIG. 1. Dominant weights of the weight system of the SU(3)-representation (3,2), their multiplicities, and the positive roots by which they differ.

$$
\begin{array}{lll}
|10\rangle_j, j=1,2,3, & R_2|\bar{1}1\rangle_j = R_2R_1|10\rangle_j = |0\bar{1}\rangle_j, & \\
R_1|10\rangle_j = |\bar{1}1\rangle_j, & & (6.8)\\
|21\rangle_i,\ i=1,2, & R_2|\bar{2}3\rangle_i = R_2R_1|21\rangle_i = |1\bar{3}\rangle_i, & \\
R_1|21\rangle_i = |\bar{2}3\rangle_i, & R_1|3\bar{1}\rangle_i = R_1R_2|21\rangle_i = |\bar{3}2\rangle_i, & \\
R_2|21\rangle_i = |3\bar{1}\rangle_i. & R_1|1\bar{3}\rangle_i = R_1R_2R_1|21\rangle_i = R_2|\bar{3}2\rangle_i & \\
 & \qquad = R_2R_1R_2|21\rangle_i = |\bar{1}\bar{2}\rangle_i. &
\end{array}
$$

The last example, although it still refers only to a group of rank 2, makes it obvious that it is impractical to explicitly write all basis vectors in larger spaces. Instead one should construct the basis for dominant weight subspaces and any other ones only when needed for a particular task at hand.

Example 5 [O(16)-representation $(0000000{}^1_1)$ of dim = 11 440]: The dominant weights and their multiplicities are found in Ref. 18; they are shown on Fig. 2. Assuming that an orthogonal basis in each of the three dominant weight subspaces of dim > 1 has been constructed, the rest of the construction is a mechanical application of CCO similar to (6.8). Thus from $|000000{}^1_1\rangle$ one gets $2^7{\cdot}8!/7! = 1024$ other basis vectors, from each of the three $|000000{}^1_1\rangle_j$, $j = 1,2,3$, one gets $2^7{\cdot}8!/5!{\cdot}4! = 1792$ others, from each of $|00100{}^0_0\rangle_k$, $k = 1,\ldots,10$, one gets $2^6{\cdot}8!/3!{\cdot}2^4{\cdot}5! = 448$ new ones, and from each $|100000{}^0_0\rangle_n$, $n = 1,\ldots,35$, one gets $2^7{\cdot}8!/2^6{\cdot}7! = 16$ new basis vectors. Here the numbers are calculated using (4.28) and the orders of Weyl groups given for instance in Refs. 15 or 18.

Suppose now that we need the basis in a particular weight subspace, say $V(\lambda) = V(01\bar{1}0\bar{1}0{}^1_{\bar{1}})$. Applying R_i with subscripts corresponding to negative entries in the corresponding weight for as long as possible one finds

$$V(000010{}^0_0) = R_5R_3R_7R_6R_4R_5V(01\bar{1}0\bar{1}0{}^1_{\bar{1}})\,. \qquad (6.9)$$

Then applying R_i^{-1} in the inverse order to that in (6.9) to the basis vectors $|000010{}^0_0\rangle_i$, $i = 1,2,3$, one gets the desired basis. Arrived at in this way the element $\widetilde{w}$ of N is bound to be of minimal type.

Let us now exemplify the truly difficult part of our problem: construction of bases in dominant weight subspaces. Also here CCO operators provide a valuable tool.

Example 6 [the adjoint representation (1,1) of SU(3)]: The dominant weight subspace $V(00)$ is of dim = 2. Its basis can be constructed as follows: Consider the highest irreducible component in the tensor product $(1,0) \otimes (0,1)$ which is the adjoint representation. Its dominant weight vectors $|11\rangle$, $|00\rangle_A$, and $|00\rangle_B$ can be chosen as

$$
\begin{aligned}
&|11\rangle = |10\rangle|01\rangle,\\
&|00\rangle_A = (1/\sqrt{2})\,(|\bar{1}1\rangle|1\bar{1}\rangle + |10\rangle|\bar{1}0\rangle), \qquad (6.10)\\
&|00_B = (1/\sqrt{2})(|\bar{1}1\rangle|1\bar{1}\rangle + |0\bar{1}\rangle|01\rangle)\,.
\end{aligned}
$$

Here the linearly independent but nonorthogonal vectors $|00\rangle_A$ and $|00\rangle_B$ span $V(00)$. Instead of (6.10) one could observe[19] that N acts irreducibly on $V(00)$. Its two-dimensional representation is generated by matrices

$$m_1 = \begin{pmatrix}\bar{1} & 0\\ 0 & 1\end{pmatrix}, \quad m_2 = \frac{1}{2}\begin{pmatrix}\bar{1} & \sqrt{3}\\ \sqrt{3} & \bar{1}\end{pmatrix} \qquad (6.11)$$

(cf. Table IX, Ref. 16) and, by definition of N, also by R_1 and R_2. On $V(00)$ one can identify m_1 with R_1. Then the eigenvectors R_1

$$R_1|00\rangle_\pm = \pm|00\rangle_\pm \qquad (6.12)$$

provide an orthogonal basis of $V(00)$. Using (6.10) one has explicitly

$$
\begin{aligned}
&|00\rangle_- = (1/\sqrt{2})(|\bar{1}1\rangle|1\bar{1}\rangle + |10\rangle|\bar{1}0\rangle),\\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad (6.13)\\
&|00\rangle_+ = (1/\sqrt{6})(|\bar{1}1\rangle|1\bar{1}\rangle - |10\rangle|\bar{1}0\rangle + 2|0\bar{1}\rangle|01\rangle).
\end{aligned}
$$

Example 7: As our next example let us construct the dominant weight basis vectors (6.7). For that consider the representation (3,2) as the highest irreducible component in $(3,0) \otimes (0,2)$. Since (3,0) and (0,2) have only one-dimensional weight spaces, their weight vectors $|{}^{30}_{\lambda}\rangle$ and $|{}^{02}_{\mu}\rangle$ provide a basis for our problem. For λ^+ of multiplicity one in Ω of (3,2), one has

$$|{}^{32}_{32}\rangle = |{}^{30}_{30}\rangle|{}^{02}_{02}\rangle,\ |{}^{32}_{13}\rangle = |{}^{30}_{11}\rangle|{}^{02}_{02}\rangle,\ |{}^{32}_{40}\rangle = |{}^{30}_{30}\rangle|{}^{02}_{10}\rangle. \qquad (6.14)$$

The two-dimensional subspace $V(02)$ is stabilized by R_1,

$$R_1V(02) = V(02)\,. \qquad (6.15)$$

Hence, its basis can be taken as eigenvectors of R_1. In order to identify the eigenvalues, it suffices to notice that there are two α_1-strings passing through $V(02)$ of V corresponding to $SU^{\alpha_1}(2)$ representations of dimensions 5 and 3. Since according to (2.17), $R\,|{}^4_0\rangle = |{}^4_0\rangle$ but $R\,|{}^2_0\rangle = -|{}^2_0\rangle$, the R_1-eigenvalues are ± 1. Consequently, we can choose

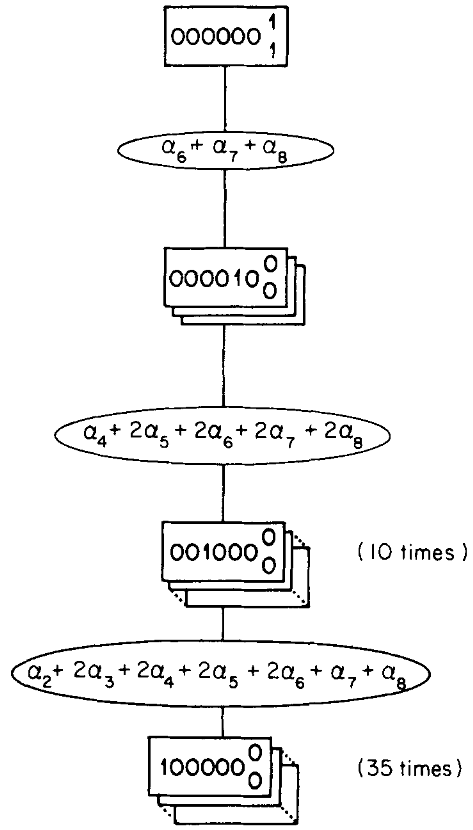

FIG. 2. Dominant weights of the representation $(000000{}^1_1)$ of O(16) of dimension 11440, their multiplicities, and the positive roots by which they differ.

$$R_1|{}^{32}_{02}\rangle_1 = |{}^{32}_{02}\rangle_1 \quad \text{and } R_1|{}^{32}_{02}\rangle_2 = -|{}^{32}_{02}\rangle_2 \tag{6.16}$$

as the basis $V(02)$, or explicitly

$$\begin{aligned} |{}^{32}_{02}\rangle_1 &= (1/\sqrt{2})(|{}^{32}_{\bar{1}2}\rangle|{}^{02}_{10}\rangle + |{}^{30}_{11}\rangle|{}^{02}_{\bar{1}1}\rangle), \\ |{}^{32}_{02}\rangle_2 &= (1/\sqrt{6})(|{}^{30}_{\bar{1}2}\rangle|{}^{02}_{10}\rangle - |{}^{30}_{11}\rangle|{}^{02}_{\bar{1}1}\rangle + 2|{}^{30}_{00}\rangle|{}^{02}_{02}\rangle)\,. \end{aligned} \tag{6.17}$$

Then it is natural to also choose

$$\begin{aligned} |{}^{32}_{21}\rangle_1 &= (1\sqrt{6})\, e_1|{}^{32}_{02}\rangle_1 = \tfrac{1}{2}(\sqrt{3}\,|{}^{30}_{11}\rangle|{}^{02}_{10}\rangle + |{}^{30}_{30}\rangle|{}^{02}_{\bar{1}1}\rangle), \\ |{}^{32}_{21}\rangle_2 &= (1/\sqrt{2})\, e_1|{}^{32}_{02}\rangle_2 = (1\sqrt{12})(|{}^{30}_{11}\rangle|{}^{02}_{10}\rangle \\ &\qquad - \sqrt{3}|{}^{30}_{30}\rangle|{}^{02}_{\bar{1}1}\rangle + 2\sqrt{2}|{}^{30}_{2\bar{1}}\rangle|{}^{02}_{02}\rangle)\,. \end{aligned} \tag{6.18}$$

The coefficients above (other than the overall normalization) are a result of the application of $SU^{\beta}(2)$ generators along the corresponding β-string [cf. (3.14)] according to (2.5). The three-dimensional subspace $V(10)$ is stabilized by the subgroup $\langle R_2, R_1^2\rangle$ of N generated by R_2 and R_1^2. There are three α_2-strings passing through it of lengths 5, 2, and 1. Due to (2.17), one can thus require that

$$\begin{aligned} &R_2|{}^{32}_{10}\rangle_1 = |{}^{32}_{10}\rangle_1 \text{ and } e_2^2|{}^{32}_{10}\rangle_1 \sim |{}^{32}_{\bar{1}4}\rangle = |{}^{30}_{\bar{1}2}\rangle|{}^{02}_{02}\rangle = R_1|{}^{32}_{13}\rangle, \\ &R_2|{}^{32}_{10}\rangle_2 = -|{}^{32}_{10}\rangle_2 \text{ and } e_2|{}^{32}_{10}\rangle_2 \sim (1-R_2)\, e_{-2}|{}^{32}_{02}\rangle_2, \\ &R_2|{}^{32}_{10}\rangle_3 = |{}^{32}_{10}\rangle_3 \text{ and } e_{\pm 2}|{}^{32}_{10}\rangle_3 = 0, \end{aligned} \tag{6.19}$$

where $\sim$ indicates that both sides differ only by a constant nonzero factor.

Explicitly (6.19) is

$$\begin{aligned} |{}^{32}_{10}\rangle_1 &= (1/\sqrt{6})(|{}^{30}_{1\bar{2}}\rangle|{}^{02}_{02}\rangle + |{}^{30}_{\bar{1}2}\rangle|{}^{02}_{2\bar{2}}\rangle + 2|{}^{30}_{00}\rangle|{}^{02}_{10}\rangle), \\ |{}^{32}_{10}\rangle_2 &= (1/\sqrt{6})(|{}^{30}_{1\bar{2}}\rangle|{}^{02}_{02}\rangle - |{}^{30}_{\bar{1}2}\rangle|{}^{02}_{2\bar{2}}\rangle - \sqrt{2}|{}^{30}_{2\bar{1}}\rangle|{}^{02}_{\bar{1}1}\rangle \\ &\qquad - \sqrt{2}|{}^{30}_{11}\rangle|{}^{02}_{0\bar{1}}\rangle), \\ |{}^{32}_{10}\rangle_3 &= (1/\sqrt{30})(3|{}^{30}_{11}\rangle|{}^{02}_{0\bar{1}}\rangle - 3|{}^{30}_{2\bar{1}}\rangle|{}^{02}_{\bar{1}1}\rangle + \sqrt{2}|{}^{30}_{\bar{1}2}\rangle|{}^{02}_{2\bar{2}}\rangle \\ &\qquad + \sqrt{2}|{}^{30}_{1\bar{2}}\rangle|{}^{02}_{02}\rangle - \sqrt{2}|{}^{30}_{00}\rangle|{}^{02}_{10}\rangle + \sqrt{6}|{}^{30}_{30}\rangle|{}^{02}_{\bar{2}0}\rangle). \end{aligned} \tag{6.20}$$

Let us point out that using the product form (6.20) for each $|{}^{32}_{\lambda}\rangle$, the conditions (6.19) do not guarantee that $|{}^{32}_{10}\rangle_3$ lies in V because the product space is reducible and contains other SU(3) irreducible representations than (3,2). (In fact there are six linearly independent vectors $|{}^{\Lambda}_{10}\rangle$ in the product space corresponding to several highest weights Λ.) In order to assure that $|{}^{32}_{10}\rangle_3 \subset V$, one can proceed for instance as follows. A third vector of $V(10)$ which is linearly independent from $|{}^{32}_{10}\rangle_1$ and $|{}^{32}_{10}\rangle_2$, is $(1+R_2)\, e^2_{-2}|{}^{30}_{\bar{1}2}\rangle|{}^{02}_{02}\rangle$. Then $|{}^{32}_{10}\rangle_3$ is the linear combination of the three which is normalized and orthogonal to $|{}^{32}_{10}\rangle_1$ and $|{}^{32}_{10}\rangle_2$.

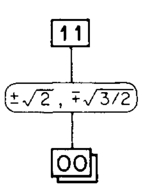

FIG. 3. Essentials of the basis for the adjoint representation of SU(3). The 1×2 matrix in the light rounded box gives the matrix elements of the generators $e_{\pm 1 \pm 2}$ between the orthonormal dominant weight vectors; upper (lower) signs refer to matrix elements of $e_{-1-2}(e_{1+2})$.

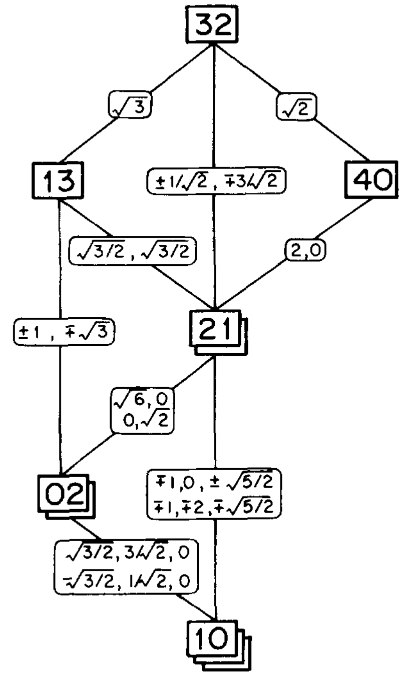

FIG. 4. Essentials of the basis for the SU(3) representation (3,2): dominant weights, their multiplicities and matrix elements of generators between orthonormal basis vectors of dominant weight subspaces. Upper (lower) signs correspond to lowering (raising) generators.

Finally, let us illustrate how all the relevant information concerning a basis and corresponding matrix elements can be presented and used [problem (ii)(b) of the Introduction].

Example 8: Let us continue Example 6. In Fig. 3 one finds the dominant weights of the adjoint representation of SU(3) and two matrices representing the action of the generator $e_{1+2} = e_1e_2 - e_2e_1$ on the chosen basis (6.13) of $V(00)$ and $e_{-1-2} = e_{-1}e_{-2} - e_{-2}e_{-1}$ on $V(11)$. Namely,

$$e_{-1-2}|11\rangle = \sqrt{1/2}\,|00\rangle_- - \sqrt{3/2}\,|00\rangle_+\,, \tag{6.21}$$

$$e_{1+2}|00\rangle_- = \sqrt{1/2}|11\rangle, \quad e_{1+2}|00\rangle_+ = \sqrt{3/2}|11\rangle\,. \tag{6.22}$$

Using Fig. 3 many other matrix elements can readily be found, for instance,

$$\begin{aligned} e_{-2}|\bar{1}2\rangle &= e_{-2}R_1|11\rangle = R_1R_2^{-1}e_{-2}R_1|11\rangle \\ &= -R_1e_{-1-2}|11\rangle \\ &= -R_1(\sqrt{1/2}|00\rangle_- - \sqrt{3/2}|00\rangle_+)\,. \end{aligned} \tag{6.23}$$

Similarly, using (6.12) and (6.23), one has

$$e_{-2}|\bar{1}2\rangle = \sqrt{1/2}|00\rangle_- + \sqrt{3/2}|00\rangle_+\,. \tag{6.24}$$

Example 9: Consider again Example 7. On Fig. 4 we have summarized the relevant information, i.e., the basis vectors (6.14), (6.17), and (6.18) together with the matrix elements of generators relating them. Thus for instance, the nonzero matrix elements of $e_{-1-2} = e_{-1}e_{-2} - e_{-2}e_{-1}$ are read off Fig. 4 as

$$e_{-1-2}|32\rangle = (1/\sqrt{2})|21\rangle_1 - (3/\sqrt{2})|21\rangle_2,$$

and

$$e_{-1-2}|21\rangle_1 = -|10\rangle_1 + \sqrt{5/2}|10\rangle_3,$$

$$e_{-1-2}|21\rangle_2 = -|10\rangle_1 - 2|10\rangle_2 - \sqrt{5/2}|10\rangle_3,$$

and

$$e_{-1-2}|13\rangle = |02\rangle_1 = \sqrt{2}|02\rangle_2,$$

while for $e_{1+2} = e_1e_2 - e_2e_1$, one has

$$e_{1+2}|10\rangle_1 = |21\rangle_1 + |21\rangle_2,$$

$$e_{1+2}|10\rangle_2 = 2|21\rangle_2,$$

$$e_{1+2}|10\rangle_3 = -\sqrt{5/2}|21\rangle_1 + \sqrt{5/2}|21\rangle_2,$$

$$e_{1+2}|21\rangle_1 = -(1/\sqrt{2})|32\rangle,$$

$$e_{1+2}|21\rangle_1 = (3/\sqrt{2})|21\rangle_2,$$

$$e_{1+2}|02\rangle_1 = -|13\rangle, \quad e_{1+2}|02\rangle_2 = \sqrt{3}|13\rangle.$$

Proceeding as in the previous example one finds any other matrix elements of e_β, $\beta \in \Delta$, in terms of those of Fig. 4.

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